



Fluid Dynamics and Transport Phenomena

Hydrodynamic dispersion of reactive solute in a Hagen–Poiseuille flow of a layered liquid

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ABSTRACT

An analysis of the solute dispersion in the liquid flowing through a pipe by means of Aris–Barton's 'method of moments', under the joint effect of some finite yield stress and irreversible absorption into the wall is presented in this paper. The liquid is considered as a three-layer liquid where the center region is Casson liquid surrounded by Newtonian liquid layer. A significant change from previous modelling exercises in the study of hydrodynamic dispersion, different molecular diffusivity has been considered for the different region yet to be constant. For all time period, finite difference implicit scheme has been adopted to solve the integral moment equation arising from the unsteady convective diffusion equation. The purpose of the study is to find the dependency of solute transport coefficients on absorption parameter, yield stress, viscosity ratio, peripheral layer variation and in addition with various diffusivity coefficients in different liquid layers. This kind of study may be useful for understanding the dispersion process in the blood flow analysis.

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1. Introduction

Dispersion is the process which is utilized as an efficient means to accomplish the dilution or mixing. In recent days, the study on the longitudinal dispersion has gained attention due to its practical importance in the field of chemical engineering, physiological fluid dynamics, environmental sciences, biomedical engineering *etc.* for both corporeal and extra-corporeal. Specifically nutrients, metabolic items, drugs, *etc.* get transported through blood flow, as a consequence of diffusive and convective mechanisms. The pioneer researcher in this field was Taylor [1,2] and a vital resulting commitment was tailored by Aris [3], who framed and partially solved equations for the integral moments of the cloud of contaminant. Sankarasubramanian and Gill [4] developed a derivative expansion method *viz.* 'generalized dispersion model', which holds true for large as well as small times. A numerical study was performed by Ananthakrishnan *et al.* [5] to solve the convective diffusion equation by a standard finite difference method which went on to show that the Taylor–Aris dispersion theory was pertinent only for adequately expansive estimation of the dimensionless time, *i.e.*, $t > 0.8$. Barton [6] refined Aris's model and extended the legitimacy to cover even small and moderate times.

Following different techniques, a large number of articles [7–12] have been found in literature dealing with longitudinal dispersion in

steady and unsteady flows within the Newtonian and non-Newtonian fluids (pipe/channels) with or without chemical reactions. Aris [13] first examined the longitudinal dispersion in an infinite tube by considering the periodic pressure gradient as the driving force. A solution for the problem of transient multicomponent dispersion in fully developed laminar flow was determined by Harry [14]. The unsteady convective diffusion with a heterogeneous chemical reaction in a plane-Poiseuille flow of micropolar fluid was studied by Siddheshwar and Manjunath [15]. Using asymptotic analysis with respect to the pipe thickness, the transport of reactive solute by diffusion and convection in a thin (or long) curved and circular pipe was considered by Pažanin [16,17]. For making a rise in solute diffusion, Duijn *et al.* [18] proposed a model of the reactive flow through a pore, under dominant Peclet and Damkohler numbers. Paul [19], and Bandyopadhyay and Mazumder [20] studied the effect of unsteady flow in solute dispersion considering the movement of the boundary wall. Using the method of moments Bandyopadhyay and Mazumder [21] studied the effect of pulsatile flow on the dispersion process in a channel, also Sarkar and Jayaraman [22], Mazumder and Mondal [23] investigated the effect of wall absorption on dispersion in oscillatory flow in an annulus and explained the application of their study to a catheterized artery.

In modern days, dispersion in non-Newtonian fluids is being widely used in the field of bio-chemical processing, cardiovascular system, polymer processing *etc.* In 1993, Sharp [24] studied the dispersion phenomena in non-Newtonian fluids (Casson, Bingham plastic and power law fluids) through conduits using the Taylor–Aris dispersion model. A

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brief review on the dispersion in power law fluids was done by Agrawal and Jayaraman [25]. Using the generalized dispersion model of Sankarasubramanian and Gill [4], shear augmented unsteady dispersion of a solute in Casson fluid flowing through a conduit was analyzed by Dash *et al.* [26]. The study revealed that in blood flow the yield stress considerably affects dispersion rate of fluid. The influence of interphase mass transfer on solute dispersion in a channel and tube flow of power-law fluid are inspected by Siddheshwar *et al.* [27,28] after the injection of the solute. Lately, Nagarani *et al.* [29] discussed the longitudinal dispersion and its application in catheterized artery assuming the Casson fluid through an annulus.

In some of the above investigations, the effectiveness of longitudinal dispersion in blood flow was discussed where blood has been treated as Newtonian and non-Newtonian liquid relying upon the estimation of shear rates. The suspicion of Newtonian behavior of blood is adequate for high shear rate flow through larger arteries [30]. Yet, blood, being a suspension of cells in plasma, exhibits non-Newtonian behavior at low shear rate $<10\text{s}^{-1}$ in small diameter arteries (0.02–0.1 mm) [31]. Different experiments were performed on blood (Scott Blair [32], Charm and Kurland [33]) with the assortment of hematocrits, anticoagulants, temperature and so on suggested that the behavior of blood at low shear rate can be described sufficiently by the Casson model. In particular, when blood flows through small blood vessels, the presence of a peripheral layer of plasma (Newtonian liquid) and a core region of suspension of all the erythrocytes as a non-Newtonian liquid has been found, which was experimentally appeared by Bugliarello and Sevilla [34] and Cokelet [35]. Thus, for a practical portrayal of the blood stream, it is more appropriate to regard blood as a two-liquid (or three-layer liquid) model comprising all the erythrocytes thought to be Casson liquid and a peripheral layer of plasma as Newtonian liquid. There are several works available in the literature revealing the significance of the peripheral layer in the functioning of the flow characteristics of the arterial system. Srivastava and Saxena [36] showed that the resistance to flow increases as yield stress increases and also the magnitudes of the flow resistance found to be decreasing with the increase in the expansion of the peripheral layer. In a two-fluid, blood flow analysis, Sankar [37, 38] observed that the velocity distribution and flow rate decrease but the wall shear stress, width of the plug flow region and longitudinal impedance increase when the yield stress increases. We have seen very few works of dispersion considering peripheral layer effects, such as Shukla and Parihar [39], and Shukla and Gupta [40]. To the best of our knowledge, there is hardly any work in the literature considering the effect of the peripheral layer on dispersion process, when the core region is Casson liquid and a peripheral layer of plasma is Newtonian liquid. Moreover, the present study manages the irreversible absorption at the boundary of the pipe. Due to the fact of peripheral layer presence, the viscosity ratio appears for the first time in the study of hydrodynamic dispersion. The present model may be useful for the physiological blood flow analysis as the nature of the effects considered in the study, the work is much closer to that in a physiological situation compared to several other works in literature.

2. Mathematical Formulation

A uni-directional, steady laminar, axi-symmetric, fully-developed flow of an incompressible, three-layer liquid through a circular pipe of radius R is considered. It is assumed that the core region is Casson liquid enclosed by a thin layer of Newtonian liquid.

Fig. 1 shows the flow geometry with a cylindrical coordinate system where the axial and radial coordinates are represented by $\bar{z}$ and $\bar{r}$ respectively (the bar denotes dimensional quantities). On account of enough axial covering of the tube and for the low Reynolds number flow, it can be shown that the radial velocity is negligibly small and can be disregarded, thus the fully developed flow is only in the axial direction. Because of the axi-symmetry all quantities are independent of θ .

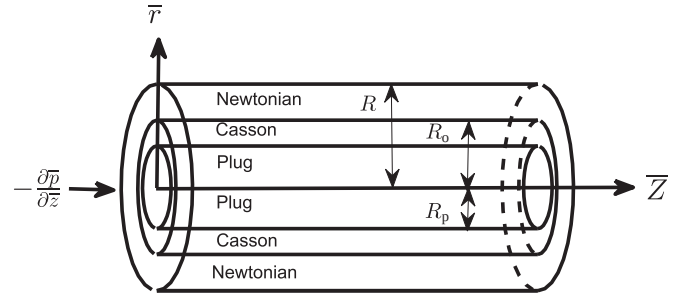


Fig. 1. Schematic diagram of the setup under consideration.

The governing equations of motion for the axial flow are given by

$$-\frac{\partial \bar{p}}{\partial \bar{z}} = \frac{1}{\bar{r}} \frac{\partial (\bar{r} \bar{\tau}_c)}{\partial \bar{r}} \quad 0 \leq \bar{r} \leq \bar{R}_o, \quad (1)$$

$$-\frac{\partial \bar{p}}{\partial \bar{z}} = \frac{1}{\bar{r}} \frac{\partial (\bar{r} \bar{\tau}_n)}{\partial \bar{r}} \quad \bar{R}_o \leq \bar{r} \leq \bar{R}. \quad (2)$$

The shear stress in the Casson region $\mathcal{R}_c (= [0, \bar{R}_o])$ and Newtonian region $\mathcal{R}_n (= [\bar{R}_o, \bar{R}])$ are given by the constitutive relations:

$$\left. \begin{aligned} \bar{\tau}_c^{\frac{1}{2}} &= \bar{\tau}_y^{\frac{1}{2}} + \left(-\bar{\mu}_c \frac{\partial \bar{u}_c}{\partial \bar{r}} \right)^{\frac{1}{2}}, & \bar{\tau}_c &\geq \bar{\tau}_y \text{ for } \bar{R}_p \leq \bar{r} \leq \bar{R}_o, \\ \frac{\partial \bar{u}_c}{\partial \bar{r}} &= 0, & \bar{\tau}_c &\leq \bar{\tau}_y \text{ for } 0 \leq \bar{r} \leq \bar{R}_p, \\ \bar{\tau}_n &= -\bar{\mu}_n \frac{\partial \bar{u}_n}{\partial \bar{r}}, & \bar{\tau}_y &= 0 \text{ for } \bar{R}_o \leq \bar{r} \leq \bar{R}, \end{aligned} \right\} \quad (3)$$

where $\bar{\tau}_y$ is the yield stress. In Eq. (3) whenever $\bar{\tau}_c \leq \bar{\tau}_y$, the velocity gradient will be zero and as a consequence, plug flow region $\mathcal{R}_p (= [0, \bar{R}_p])$ appears within $\mathcal{R}_c$.

The boundary, symmetry and matching conditions for solving Eqs. (1)–(3) are:

$$\left. \begin{aligned} \bar{\tau}_c &\text{ is finite and } \frac{\partial \bar{u}_c}{\partial \bar{r}} = 0 & \text{ at } \bar{r} = 0, \\ \bar{\tau}_c = \bar{\tau}_n &\text{ and } \bar{u}_c = \bar{u}_n & \text{ at } \bar{r} = \bar{R}_o, \\ \bar{u}_n &= 0 & \text{ at } \bar{r} = \bar{R}, \end{aligned} \right\} \quad (4)$$

Let us now consider the transport and spreading of chemical species with the solvent flowing through the tube which is supposed to be completely miscible with the liquid. The species are alleged to contain irreversible absorption into the wall.

In the present scenario, the transport equation that governs $\bar{C}(\bar{r}, \bar{z})$ is the unsteady convective diffusion equation and is given by

$$\frac{\partial \bar{C}}{\partial \bar{t}} + \bar{u}(\bar{r}) \frac{\partial \bar{C}}{\partial \bar{z}} = \frac{\bar{D}}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left(\bar{r} \frac{\partial \bar{C}}{\partial \bar{r}} \right) + \bar{D} \frac{\partial^2 \bar{C}}{\partial \bar{z}^2}, \quad (5)$$

where $\bar{D}$ is the molecular diffusivity and we consider $\bar{D}$ to be different, yet constant in the three regions $\mathcal{R}_p$, $\mathcal{R}_c$ and $\mathcal{R}_n$. So we assume

$$\bar{D} = \begin{cases} \bar{D}_p & \text{in } \mathcal{R}_p \\ \bar{D}_c & \text{in } \mathcal{R}_c \\ \bar{D}_n & \text{in } \mathcal{R}_n \end{cases} \quad (6)$$

The initial and boundary conditions for the transport Eq. (5) are considered as

$$\bar{C}(0, \bar{r}, \bar{z}) = \bar{C}_0 \quad B(\bar{r})\psi(\bar{z}), \quad 0 < \bar{r} < \bar{R} \quad (7)$$

$$\frac{\partial \bar{C}}{\partial \bar{r}} = 0 \quad \text{at} \quad \bar{r} = 0, \quad (8)$$

$$\frac{\partial \bar{C}}{\partial \bar{r}} + \beta \bar{C} = 0 \quad \text{at} \quad \bar{r} = \bar{R}, \quad (9)$$

where $B(\bar{r})$ and $\psi(\bar{z})$ are needed to be specified to solve Eq. (5), $\bar{C}_0$ is the initial concentration of the slug input. The absorbing boundary condition at the wall of the tube is represented by Eq. (9).

Using the following dimensionless quantities:

$$\left. \begin{aligned} t &= \frac{\bar{D}_n \bar{t}}{\bar{R}^2}, \quad r = \frac{\bar{r}}{\bar{R}}, \quad z = \frac{\bar{D}_n \bar{z}}{\bar{R}^2 u_0}, \quad u_0 = -\frac{\bar{R}^2}{4\bar{\mu}_n} \frac{dp}{dz}, \quad u_c = \frac{\bar{u}_c}{u_0}, \quad u_n = \frac{\bar{u}_n}{u_0}, \quad C = \frac{\bar{C}}{\bar{C}_0}, \\ \tau_c &= \frac{\bar{\tau}_c}{\bar{\mu}_n \left(\frac{u_0}{\bar{R}}\right)}, \quad \tau_n = \frac{\bar{\tau}_n}{\bar{\mu}_n \left(\frac{u_0}{\bar{R}}\right)}, \quad R_0 = \frac{\bar{R}_0}{\bar{R}}, \quad R_p = \frac{\bar{R}_p}{\bar{R}}, \quad \tau_y = \frac{\bar{\tau}_y}{\bar{\mu}_n \left(\frac{u_0}{\bar{R}}\right)}, \quad \beta = \beta \bar{R}, \\ Pe &= \frac{u_0 \bar{R}}{\bar{D}_n}, \quad m = \frac{\bar{\mu}_c}{\bar{\mu}_n}, \quad D^* = \frac{\bar{D}}{\bar{D}_n}, \end{aligned} \right\} \quad (10)$$

Eqs. (5)–(9), may be written as:

$$\frac{\partial C}{\partial t} + u(r) \frac{\partial C}{\partial z} = \frac{D^*}{r} \frac{\partial}{\partial r} \left(r \frac{\partial C}{\partial r} \right) + \frac{D^*}{Pe^2} \frac{\partial^2 C}{\partial z^2}, \quad (11)$$

$$D^* = \begin{cases} D_p & \text{in } \mathcal{R}_p \\ D_c & \text{in } \mathcal{R}_c \\ 1 & \text{in } \mathcal{R}_n \end{cases}, \quad (12)$$

along with initial and boundary conditions as:

$$C(0, r, z) = B(r)\psi(z), \quad (0 < r < 1) \quad (13)$$

$$\frac{\partial C}{\partial r} = 0 \quad \text{at} \quad r = 0, \quad (14)$$

$$\frac{\partial C}{\partial r} = -\beta C \quad \text{at} \quad r = 1. \quad (15)$$

Here u_0 is the time-averaged axial velocity, β is the absorption parameter or first order reaction rate representing the rate of loss on the wall. Pe is the Peclet number which measures the relative characteristic time of the diffusion process $\left(\frac{R^2}{D}\right)$ to the convection process $\left(\frac{R}{u_0}\right)$. Here m is the viscosity ratio which indicates the influence of Casson viscosity (μ_c) compared to the Newtonian viscosity (μ_n). The radius of the cylinder containing solute to that of the entire tube is assumed as value one, subsequently, the slug initially occupies the entire cross-section of the tube.

With the assistance of above boundary conditions (4) in dimensionless form, the velocity distributions are acquired as:

$$\begin{aligned} u_p(r) &= (1 - R_0^2) + \frac{1}{m} \left\{ (R_0^2 - R_p^2) - \frac{4\sqrt{2\tau_y}}{3} (R_0^3 - R_p^3) + \tau_y (R_0 - R_p) \right\} & 0 \leq r \leq R_p, \\ u_c(r) &= (1 - R_0^2) + \frac{1}{m} \left\{ (R_0^2 - r^2) - \frac{4\sqrt{2\tau_y}}{3} (R_0^3 - r^3) + \tau_y (R_0 - r) \right\} & R_p \leq r \leq R_0, \\ u_n(r) &= (1 - r^2) & R_0 \leq r \leq 1. \end{aligned} \quad (16)$$

where u_c and u_n are respectively the velocities for shear flow in Casson and Newtonian liquid. The constant velocity u_p appears only in the plug flow region having radius $R_p = \frac{\tau_y}{2}$. Also $R_0 (\leq 1)$ is the ratio of the central core radius and normal pipe radius.

We now consider the following two all-time approaches in studying dispersion of the reactive solute:

- (1) Aris–Barton approach and
- (2) Sankarasubramanian–Gill approach.

The two approaches are described in necessary detail in the next two sections.

3. Aris–Barton Approach

The p th moment of the distribution of the solute in the direction of flow can be described by Aris–Barton [6] as,

$$C^{(p)}(t, r) = \int_{-\infty}^{+\infty} z^p C(t, r, z) dz. \quad (17)$$

So using Eq. (17), the diffusion Eq. (11) subject to the initial and boundary conditions ((13)–(15)) can be written as

$$\frac{\partial C^{(p)}}{\partial t} - \frac{D^*}{r} \frac{\partial}{\partial r} \left(r \frac{\partial C^{(p)}}{\partial r} \right) = p u(r) C^{(p-1)} + \frac{D^*}{Pe^2} p(p-1) C^{(p-2)}, \quad (18)$$

with

$$\left. \begin{aligned} C^{(p)}(0, r) &= \begin{cases} \frac{1}{Pe} & \text{for } p = 0 \\ 0 & \text{for } p > 0 \end{cases} \\ \frac{\partial C^{(p)}}{\partial r} &= 0 \quad \text{at } r = 0, \\ \frac{\partial C^{(p)}}{\partial r} &= -\beta C^{(p)} \quad \text{at } r = 1, \end{aligned} \right\}. \quad (19)$$

The cross-sectional average of the p th moment of the distribution of solute is given by:

$$\langle C^{(p)}(t) \rangle = 2 \int_0^1 r C^{(p)}(t, r) dr, \quad (20)$$

where the angle bracket denotes the cross-sectional mean.

With this definition, Eq. (18) and the conditions (19) become,

$$\frac{d}{dt} \langle C^{(p)} \rangle + 2\beta D^* \langle C^{(p)}(t, 1) \rangle = p \langle u(r) C^{(p-1)} \rangle + \frac{D^*}{Pe^2} p(p-1) \langle C^{(p-2)} \rangle, \quad (21)$$

with the initial condition

$$\langle C^{(p)}(0) \rangle = \begin{cases} \frac{1}{Pe} & \text{for } p = 0 \\ 0 & \text{for } p > 0 \end{cases} \quad (22)$$

The p th order central moment about mean of the concentration distribution can be defined as

$$\mu_p(t) = \frac{\int_0^1 \int_0^{2\pi} \int_{-\infty}^{+\infty} r(z - z_g)^p C dr d\theta dz}{\int_0^1 \int_0^{2\pi} \int_{-\infty}^{+\infty} r C dr d\theta dz}, \quad (23)$$

where

$$z_g = \frac{\iiint z C dv}{\iiint C dv} = \frac{\langle C^{(1)} \rangle}{\langle C^{(0)} \rangle},$$

represents the ‘centroid’ or ‘first moment’ of the distribution of solute and $\langle C^{(0)} \rangle$ is the total mass of the chemical species in the flowing stream.

The expressions for central moments can be obtained from Eq. (23) as,

$$\left. \begin{aligned} \mu_2(t) &= \frac{\langle C^{(2)} \rangle}{\langle C^{(0)} \rangle} - z_g^2, \\ \mu_3(t) &= \frac{\langle C^{(3)} \rangle}{\langle C^{(0)} \rangle} - 3z_g \mu_2 - z_g^3, \\ \mu_4(t) &= \frac{\langle C^{(4)} \rangle}{\langle C^{(0)} \rangle} - 4z_g \mu_3 - 6z_g^2 \mu_2 - z_g^4, \end{aligned} \right\}. \quad (24)$$

It was Aris [3] who found the physical essence of each of the integral moments of concentration defined by Eq. (24) while explaining the dispersion phenomena on liquid. For instance, the zeroth moment gives

the aggregate range under the distribution curve which corresponds to the total mass of the solute, the first moment measures the location of the center of gravity of the slug moving with the mean velocity of the liquid and second central moment represents the variance of the distribution about the mean position whose rate of change gives the dispersion coefficient.

The reaction coefficient or exchange coefficient M_0 can be figured from the way that the total mass decays exponentially with time according to:

$$M_0(t) = \langle C^{(0)}(0) \rangle \text{Exp} \left(\int_0^t M_0 \, dt \right). \quad (25)$$

Again the center of mass moves at a speed equivalent to the advection speed, the advection coefficient M_1 is given by,

$$M_1 = \frac{dz_g}{dt}. \quad (26)$$

Following Aris [3] the apparent dispersion coefficient, D_a , can be expressed as follows:

$$D_a(\beta, \tau_y, R_0, m, Pe, t) = \frac{1}{2} \frac{d\mu_2}{dt}. \quad (27)$$

The coefficients of skewness $\nu_2 (= \mu_3/\mu_2^{3/2})$ and kurtosis $\nu_3 (= \mu_4/\mu_2^2 - 3)$ are also imperative elements for measuring the degree of symmetry and peakedness of the concentration distribution respectively.

Having discussed in detail the Aris–Barton approach to dispersion, we now move on to the details of the Sankarasubramanian–Gill approach.

4. Sankarasubramanian–Gill Approach

Following the model developed by Sankarasubramanian and Gill [4], the concentration expands as

$$C(t, r, z) = \sum_{i=0}^{\infty} f_i(t) \frac{\partial^i C_m}{\partial z^i}, \quad (28)$$

where f_i 's are the time-dependent coefficients to be determined and the dimensionless area-average concentration C_m is given by

$$C_m = C_m(t, z) = \frac{\int_0^{2\pi} d\theta \int_0^1 r C dr}{\int_0^{2\pi} d\theta \int_0^1 r dr} = 2 \int_0^1 C r dr, \quad (29)$$

using Eq. (29) in Eq. (11) alongside Eq. (28) and also the boundary conditions (13)–(15), the following dispersion model for C_m gives

$$\frac{\partial C_m}{\partial t} = \sum_{i=0}^{\infty} M_i(t) \frac{\partial^i C_m}{\partial z^i}. \quad (30)$$

According to Sankarasubramanian and Gill [4], it will be sufficient to truncate the infinite sum in Eq. (30) after involving $M_2(t)$. Where,

$$M_0(t) = -2\beta D^* f_0(1, t), \quad (31a)$$

$$M_1(t) = -2\beta D^* f_1(1, t) - 2 \int_0^1 u(r) r f_0(r, t) dr, \quad (31b)$$

$$M_2(t) = \frac{D^*}{Pe^2} - 2\beta D^* f_2(1, t) - 2 \int_0^1 u(r) r f_1(r, t) dr. \quad (31c)$$

In light of the non-zero solute flux at the outer boundary of the wall, $M_0(t)$ arises, which corresponds to the absorption parameter. As the solute undergoes an irreversible reaction at the outer boundary, it will deplete with the time that makes negative exchange coefficient. In such a case exchange coefficient would be positive as β in Eq. (15) would take on a negative sign. $M_1(t)$ and $M_2(t)$ relate to the convective and dispersion coefficients respectively. It is necessary to mention that $M_2(t)$

is analogous to the dispersion coefficient (D_a) obtained from the Aris–Barton method of moments.

For solving the above coefficients one needs to find the corresponding functions f_k 's. So, we utilize Eq. (27) in Eq. (11) along with Eq. (30) and equating the coefficient of $\frac{\partial^k C_m}{\partial z^k}$, the following partial differential equation obtained as

$$\frac{\partial f_k}{\partial t} + \sum_{i=0}^k M_i(t) f_{k-i}(r, t) + u(r) f_{k-1} - \frac{D^*}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f_k}{\partial r} \right) - \frac{D^*}{Pe^2} f_{k-2} = 0, \quad (32)$$

where $f_{-1} = f_{-2} = 0$ and $k = 0, 1, 2$.

From Eqs. (13)–(15), the initial conditions for C_m and f_k are

$$C_m(z, 0) = 2\psi(z) \int_0^1 B(r) r dr, \quad (33)$$

$$f_k(r, 0) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k = 1, 2 \end{cases}, \quad (34)$$

and the boundary conditions for the pipe are

$$\frac{\partial f_k}{\partial r} = 0 \quad \text{at } r = 0, \quad \text{for } k = 0, 1, 2 \quad (35)$$

$$\frac{\partial f_k}{\partial r} = -\beta f_k(r, t) \quad \text{at } r = 1, \quad \text{for } k = 0, 1, 2 \quad (36)$$

also

$$C_m(t, \infty) = \frac{\partial C_m}{\partial z}(t, \infty) = 0, \quad (37)$$

and

$$\int_0^1 f_k(t, r) r dr = \frac{1}{2} \delta_{k0} \quad \text{for } k = 0, 1, 2. \quad (38)$$

5. Finite Difference of the Unsteady Convective Diffusion Equation in the Three-layer Region to Obtain Transport Coefficients

Due to the presence of complexity in the analytical solution, in this section, we have numerically computed the solution of the moment Eq. (18) with initial and boundary conditions (19). For all time periods, when the solute is injected uniformly over the cross-section of the pipe, we developed a finite difference formula in view of the Crank–Nicolson implicit scheme, where the entire width of the pipe has been divided into $(M-1)$ equal parts having length Δr . The lengths and times are represented by the grid point j and i respectively, so that the general formulas are $r_j = (j-1) \times \Delta r$ and $t_i = \Delta t \times (i-1)$. In this way, for $j=1$, M corresponds to the axis of the pipe ($r=0$) and the wall ($r=1$) of the pipe respectively, also as $i=1$ corresponds to the initial time ($t=0$), where Δt and Δr are the augmentations of t and r respectively. $C^{(p)}(i, j)$ indicates the value of $C^{(p)}$ at the i -th grid point along the t -axis and j -th grid point along r -axis. The resulting finite difference formula leads to a system of algebraic equation with a tri-diagonal coefficient matrix,

$$P_j C^{(p)}(i+1, j+1) + Q_j C^{(p)}(i+1, j) + R_j C^{(p)}(i+1, j-1) = S_j. \quad (39)$$

where P_j , Q_j , R_j and S_j are the matrix elements. The finite difference form of the initial condition is,

$$C^{(p)}(1, j) = \begin{cases} \frac{1}{Pe}, & \text{for } p = 0 \\ 0, & \text{for } p > 0 \end{cases}$$

and that of boundary conditions are

$$C^{(p)}(i+1, 0) = C^{(p)}(i+1, 2), \text{ at the axis of pipe, and}$$

$$C^{(p)}(i+1, M+1) = C^{(p)}(i+1, M-1) - 2\Delta r \beta C^{(p)}(i+1, M),$$

at the surface of the pipe wall.

A MATLAB code has been developed to solve the tri-diagonal coefficient matrix rendered from Eq. (39) by the method of the Thomas algorithm (Anderson *et al.* [41]) with the help of prescribed initial and boundary conditions. The computation steps are:

Step-1: The time independent axial velocity u is computed first from Eq. (16).

Step-2: The concentration $C^{(p)}$ is then calculated from Eq. (18) as the value of $u(r)$ at each of the grid points is known from **Step-1**.

Step-3: Finally the value of $\langle C^{(p)} \rangle$ is calculated from Eq. (20) by applying Simpson's one-third rule, using the values found from **Step-1** and **Step-2** at the corresponding grid points.

Numerical calculations have been performed for various values for absorption parameter, yield stress, peripheral layer thickness, viscosity ratio, and distinct molecular diffusivities at different layers to differentiate the individual dispersion processes. Although the present scheme is linearly stable for any finite values of $\Delta t/(\Delta r)^2$, where a mesh size ($\Delta t = 0.00001$, $\Delta r = 1/(M-1)$) and $M = 100$, gives satisfactory results for different values of parameters. In all the cases, we have taken $|\frac{\partial p}{\partial z}| = 2$ and $Pe = 10^3$. From the aforementioned spatial and temporal discretization parameters, we have guaranteed ourselves of a decent request of precision of the results while varying the time step and grid spacings. Actually, the smaller time interval with sufficiently small spatial discretizations has been utilized in order to keep the accuracy of the results.

Having numerically obtained the transport coefficients from the above two all-time approaches, we now outline the procedure of obtaining the mean solute concentration distribution in the layered liquid medium.

6. Calculation of the Non-Gaussian Mean Solute Concentration Using Hermite Polynomials

The behavior of concentration distribution might also be acquired from the information of the first four central moments of the

distribution. Utilizing these four moments, it is possible to approximate the mean axial concentration distribution $C_m(z, t)$ of tracers within the flow region with the assistance of Hermite polynomial representation for non-Gaussian curves (Mehta *et al.* [42]) and is given by,

$$C_m(z, t) = \langle C^{(0)}(t) \rangle e^{-\eta^2} \sum_{n=0}^{\infty} a_n(t) H_n(\eta), \quad (40)$$

where $\eta = (z - z_g)/\sqrt{2\mu_2}$, $z_g = \frac{\langle C^{(1)} \rangle}{\langle C^{(0)} \rangle}$ and H_i the Hermite polynomials, satisfy the recurrence relation with $H_0(\eta) = 1.0$ as

$$H_{i+1}(\eta) = 2\eta H_i(\eta) - 2i H_{i-1}(\eta), \quad i = 0, 1, 2, \dots$$

The coefficients a_i s are

$$a_0 = 1/(2\pi\mu_2)^{1/2}, \quad a_1 = a_2 = 0, \quad a_3 = 2^{1/2}a_0\mu_2/24, \quad a_4 = a_0\mu_3/96$$

Therefore, given the statistical parameters in Eq. (24), the concentration distribution can be estimated from Eq. (40) at any given location in the axial direction and time.

7. Results and Discussion

The present analysis essentially deals with the stream-wise dispersion of solute in solvent flowing through a narrow pipe under the effect of peripheral layer thickness with irreversible reaction occur at the boundary. In line with the discussions outlaid in Section 1, the importance of the effect of peripheral layer in blood flow, the values of yield stress and peripheral layer thickness assume a pivotal significance. As indicated by McDonald [43], blood possesses a yield stress of $0.04 \text{ dyn} \cdot \text{cm}^{-2}$ at a hematocrit value of 40%. In contrast to published studies in this direction, the present study has investigated the parametric effect of molecular diffusivity at different liquid layers which has helped to project the importance of viscosity ratio in the present paradigm of study. In order to establish the desired parametric effect both qualitatively and quantitatively, variables were investigated in the following ranges: (1) irreversible absorption parameter (β) from 0 to 100, (2) yield stress (τ_y) from 0 to 0.2, (3) viscosity ratio (m) ranges from 1 to 1.2 and (4) peripheral layer variation (R_o) varies from 0.86 to 1. Fig. 2 demonstrates the velocity profiles with respect to the pipe radius at different parametric values of τ_y , R_o and m .

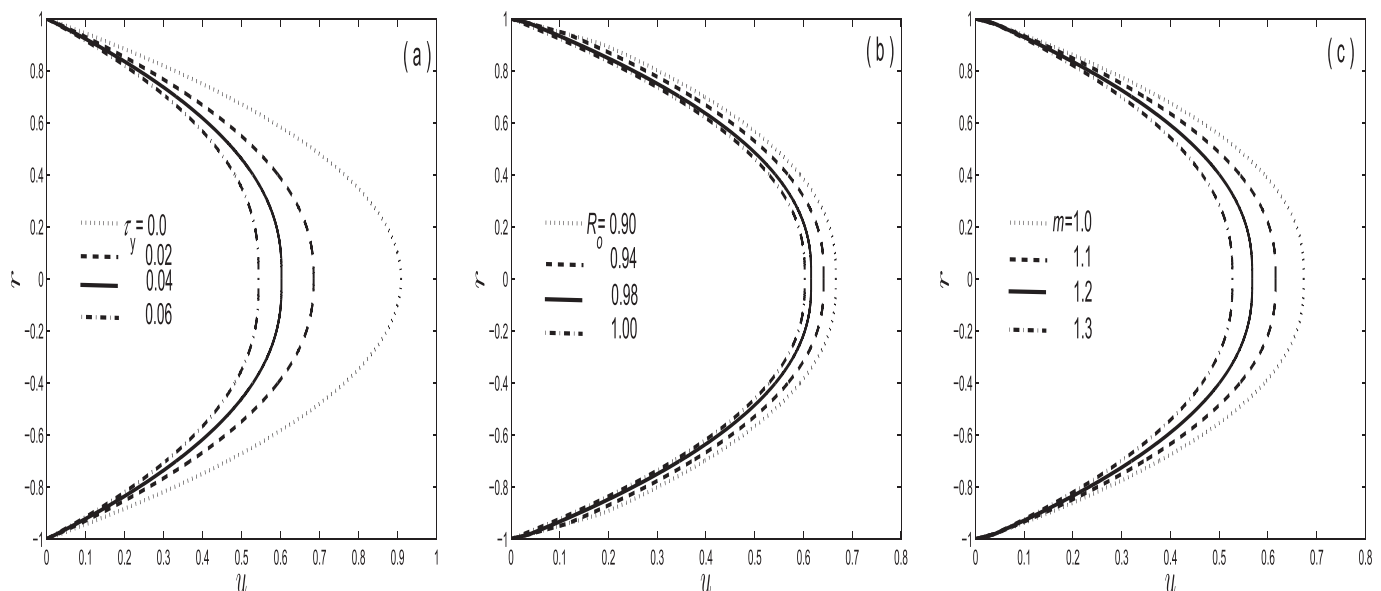


Fig. 2. Steady flow velocity profiles for different values yield stress (τ_y), peripheral layer variation (R_o) and viscosity ratio (m).

To verify the accuracy of the numerical scheme, the following cases have been considered:

- (1) In order to compare the results of dispersion obtained in the present study with that established in the study of Sankarasubramanian and Gill [4] the parameters of interphase mass transfer under constant pressure gradient have been characterized with the same values of $R_0 = 1$, $D_p = D_c = 1$, $m = 1$ and $R_p = 0.0$, (for Newtonian fluid). The present model has been further investigated with varying $R_p = 0.02, 0.04, 0.06, 0.08, 0.1, 0.2$ which corresponds to the dispersion studies of Nagarani *et al.* [44] for comparison of the model viability in the case of Casson fluid. Fig. 3 represents the plot of dispersion coefficient (M_2) vs. absorption (β). As is evident from the results obtained, the model trends completely concur qualitatively with that obtained in the studies of Sankarasubramanian and Gill [4] and with Nagarani *et al.* [44]. However the quantitative differences realized in the present study are attributed to differing normalization scales adopted in the present study.

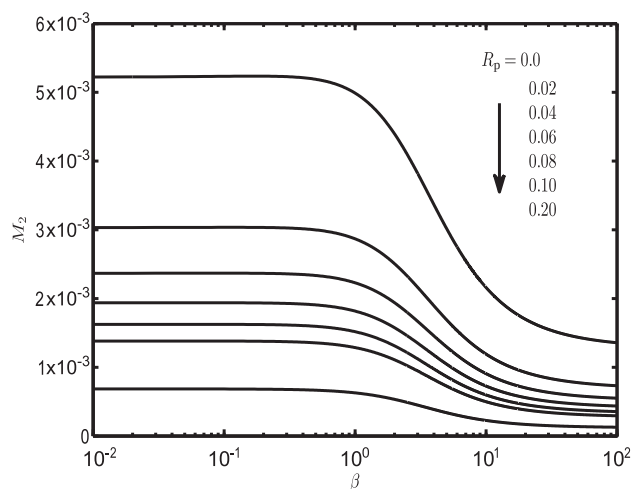


Fig. 3. Variation of dispersion coefficient (M_2) with absorption parameter (β) for various values of plug radius (R_p) for the case when $R_0 = 1$, $m = 1$ and $D_p = D_c = 1$.

- (2) In order to validate the present model response with those established in the study of Dash *et al.* [26], the simulation parameters of R_0, D_p, D_c and m bearing their individual significance had been kept at unity. To this end, the model responses outlaid in Fig. 4 display complete agreement with those obtained in the study of solute dispersion of Dash *et al.* [26] without catalytic wall reaction, i.e. $\beta = 0$ (see Fig. 4 totally coincides with Fig. 2(a) in [26]).

In order to validate the robustness of the present models, the convection diffusion mechanism paradigm was addressed, both through the method of moment [6] as well as with that of the generalized dispersion model [4]. Simulation response to this effect has been encapsulated in Fig. 5. The value of the dispersion coefficient (D_a) obtained from the 'method of moment' and the dispersion coefficient (M_2) acquired from the 'generalized dispersion model' for both Newtonian and Casson liquid case studies under the same parametric environment, yields striking quantitative and qualitative resemblances independent of the models so chosen to this end.

For the rest of our analysis, effects of different parameters on transport coefficients have been examined by Aris's method of moments.

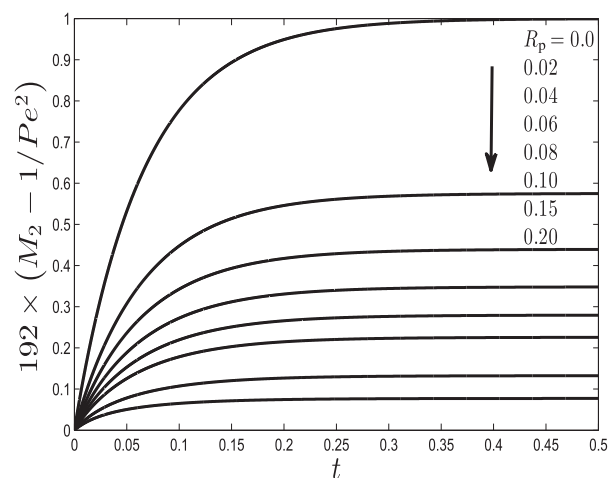


Fig. 4. Variation of dispersion coefficient [$192 \times (M_2 - 1/Pe^2)$] with time (t) for various values of plug radius (R_p) for the case when $R_0 = 1, m = 1, D_p = D_c = 1$ and $\beta = 0$.

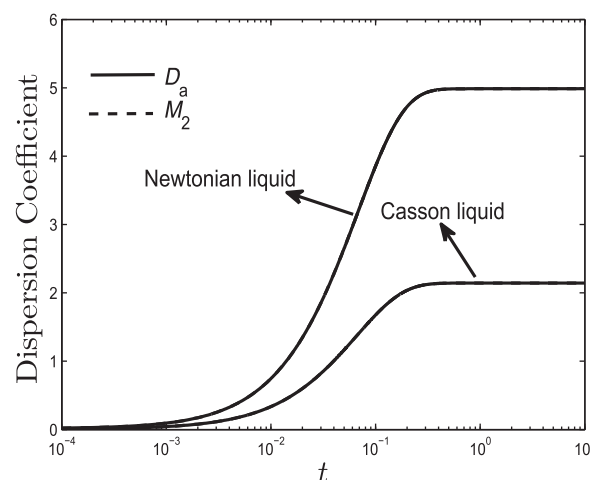


Fig. 5. Variation of dispersion coefficient with time obtained by moment method (solid line) and generalized dispersion model (dashed line). The chosen parameter values are: $m = 1$, $D_p = D_c = 1$. For the case of Newtonian liquid, $\tau_y = 0$, $R_0 = 1$ and for Casson liquid, $\tau_y = 0.04$, $R_0 = 1$.

7.1. Exchange coefficient

The variation of exchange coefficient ($M_0(t)$) with time has been interpreted through Fig. 6 under the impact of reaction parameters (β), yield stress (τ_y) and viscosity ratio (m).

Fig. 6(a) clearly shows that the increase in β makes an augmentation of M_0 . The increase in M_0 with β can be attributed to the expanding loss of solute at the wall as β increases. It is to be noted here that M_0 exists due to the presence of chemical reaction which is independent of τ_y and m as seen in Fig. 6(b, c). Eq. (31a) further elucidates that the peripheral layer effect on M_0 does not vary the outcomes acquired for purely Casson liquid flow.

7.2. Convection and dispersion coefficients

For purely Casson and peripheral layer liquid flows, variations of the convective coefficient ($M_1(t)$) and apparent dispersion coefficient ($D_a(t)$) with respect to time (t) are shown in Fig. 7 for various estimations of the reaction parameter (β), yield stress (τ_y) and viscosity ratio (m). The lines with asterisk have been shown in figures due to Casson

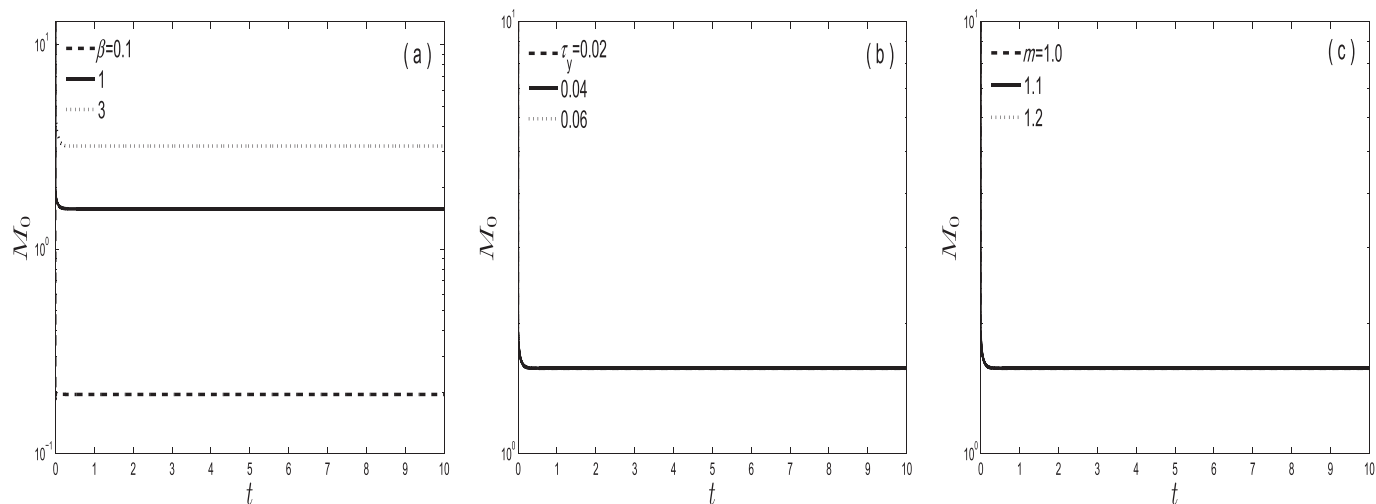


Fig. 6. Time assessment of exchange coefficients (M_0) in a purely Casson liquid flow for the fixed $R_o = D_p = D_c = 1$, (a) variation of absorption parameter (β) when $m = 1.1$ and $\tau_y = 0.04$; (b) variation of yield stress (τ_y) when $m = 1.1$ and $\beta = 1$; (c) variation of viscosity ratio (m) when $\tau_y = 0.04$ and $\beta = 1$.

liquid flow, whereas the rest are due to peripheral layer liquid. A wide range of time is considered to track the asymptotic approach of the dispersion coefficient.

Profiles in Fig. 7(a, b) represent the behavior of M_1 and D_a for same values of β variation. From Fig. 7(a, b), it is observed that M_1 increases while D_a decreases monotonically with β until it reaches their

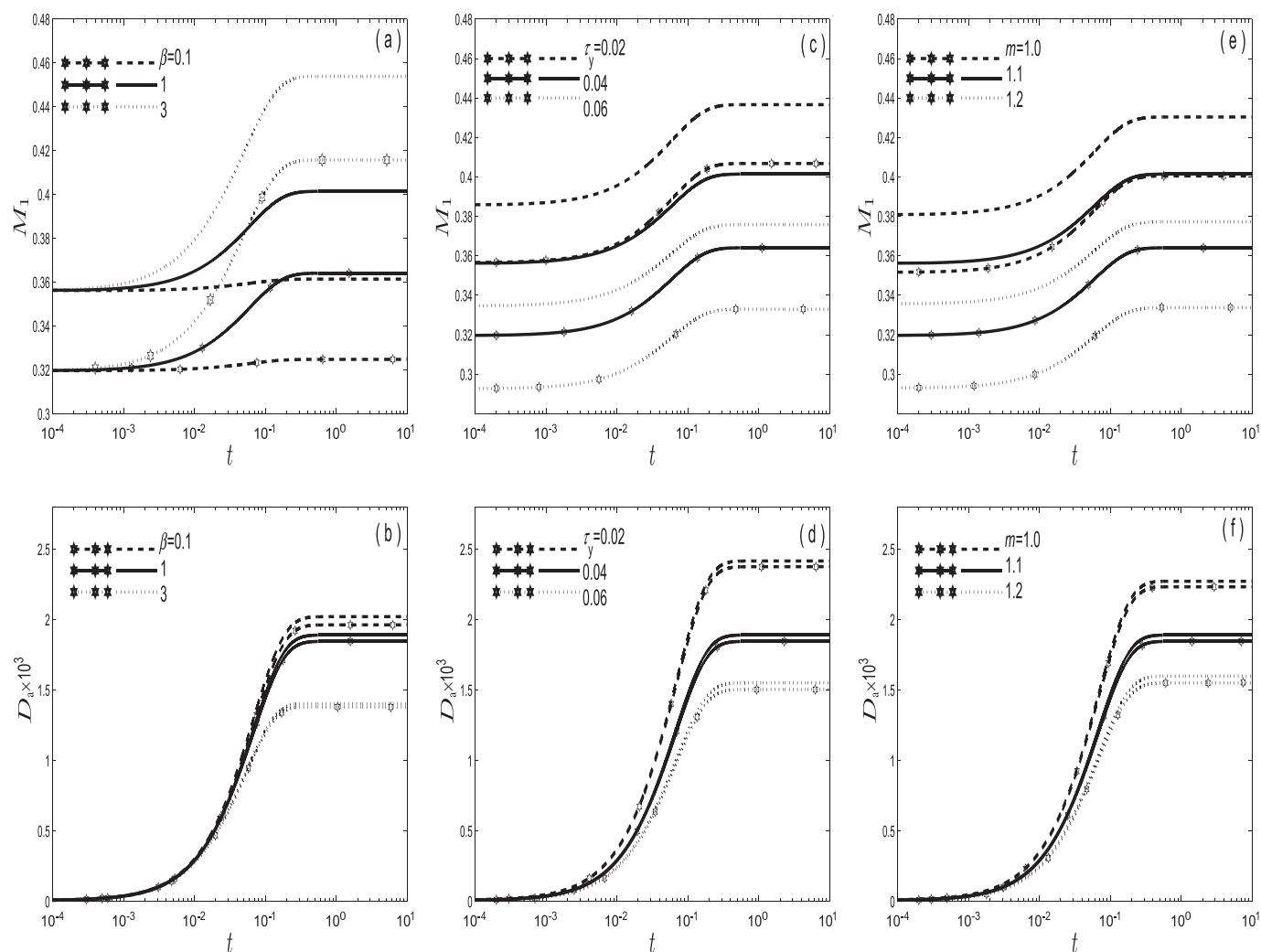


Fig. 7. Time assessment of transport coefficients (M_1 and D_a) in purely Casson liquid flow (asterisk lines) for the fixed $R_o = D_p = D_c = 1$, and in three-layer liquid flow for the fixed $R_o = 0.94$ and $D_p = D_c = 1$, (a–b) variation of absorption parameter (β) when $m = 1.1$ and $\tau_y = 0.04$; (c–d) variation of yield stress (τ_y) when $m = 1.1$ and $\beta = 1$; (e–f) variation of viscosity ratio (m) when $\tau_y = 0.04$ and $\beta = 1$.

respective steady state values. Since the wall reaction enhances the transport of solute towards the wall, subsequently M_1 increases with β . From Fig. 7(a), it can be said that M_1 reaches earlier to its asymptotic value with the decrease in β , since weak absorption makes an equilibrium between the diffusion of solute and the wall reaction occurs within small time frame. In Fig. 7(b), the physical perception about the decrement of D_a with increasing β leads to an increase in the quantity of moles of reactive material undergoing chemical reaction or absorption resulting in a change in the concentration distribution across the tube with the simultaneous consequence of a drop in dispersion coefficient. The present study yields a unique observation as evident in Fig. 7(a, b) is that the presence of a peripheral layer makes larger values in M_1 and D_a as compared to single Casson layer liquid flow. This trend is also evident from subsequent Fig. 9.

Fig. 7(c, d) has been drawn to emphasize the effect of τ_y on M_1 and D_a in order to highlight the pivotal significance of low shear rate applicable to blood flow through small blood vessels in the presence of the peripheral layer. While observing Fig. 7(c, d), it is clearly evident that M_1 and D_a decreases with increasing τ_y . This is attributed to the fact that the increase in τ_y (or in other words, an increase in the non-Newtonian parameter) decreases the velocity of the fluid. The

importance of a peripheral layer can be appreciated from the observation of the increased value of the coefficients as compared to Casson liquid alone. The reason behind these increments is due to Newtonian layer effect, which has lesser viscosity than non-Newtonian Casson liquid.

Viscosity ratio (m) is defined as the ratio between Casson viscosity (μ_c) and Newtonian viscosity (μ_n). In the present study, the appearance of viscosity ratio (m) is due to the peripheral layer. The main advantage of using m on the study of dispersion is that it directly discloses as to how strong the influence of μ_c is compared to μ_n . The time evaluation of convection coefficient (M_1) and dispersion coefficient (D_a) under the effect of m has been projected in Fig. 7(e, f). As can be inferred from Fig. 7(e, f), we can observe that M_1 and D_a increase with time (t) for a short period beyond which they attain a steady state trend. Again, as m increases the magnitudes of M_1 and D_a decrease. This happens in light of the fact that the increment of m causes a decrease in velocity distribution (see Fig. 2(c)). For Casson liquid flow, when $m = 1.1$ and $t = 0$ the increments in M_1 and D_a are 1.13 and 198.7 times the value corresponding to $t = 10$, but for a three-layer liquid this increment is 1.15 and 200 times respectively. While comparing the Casson liquid and the corresponding three-layer liquid stream, it is observed that,

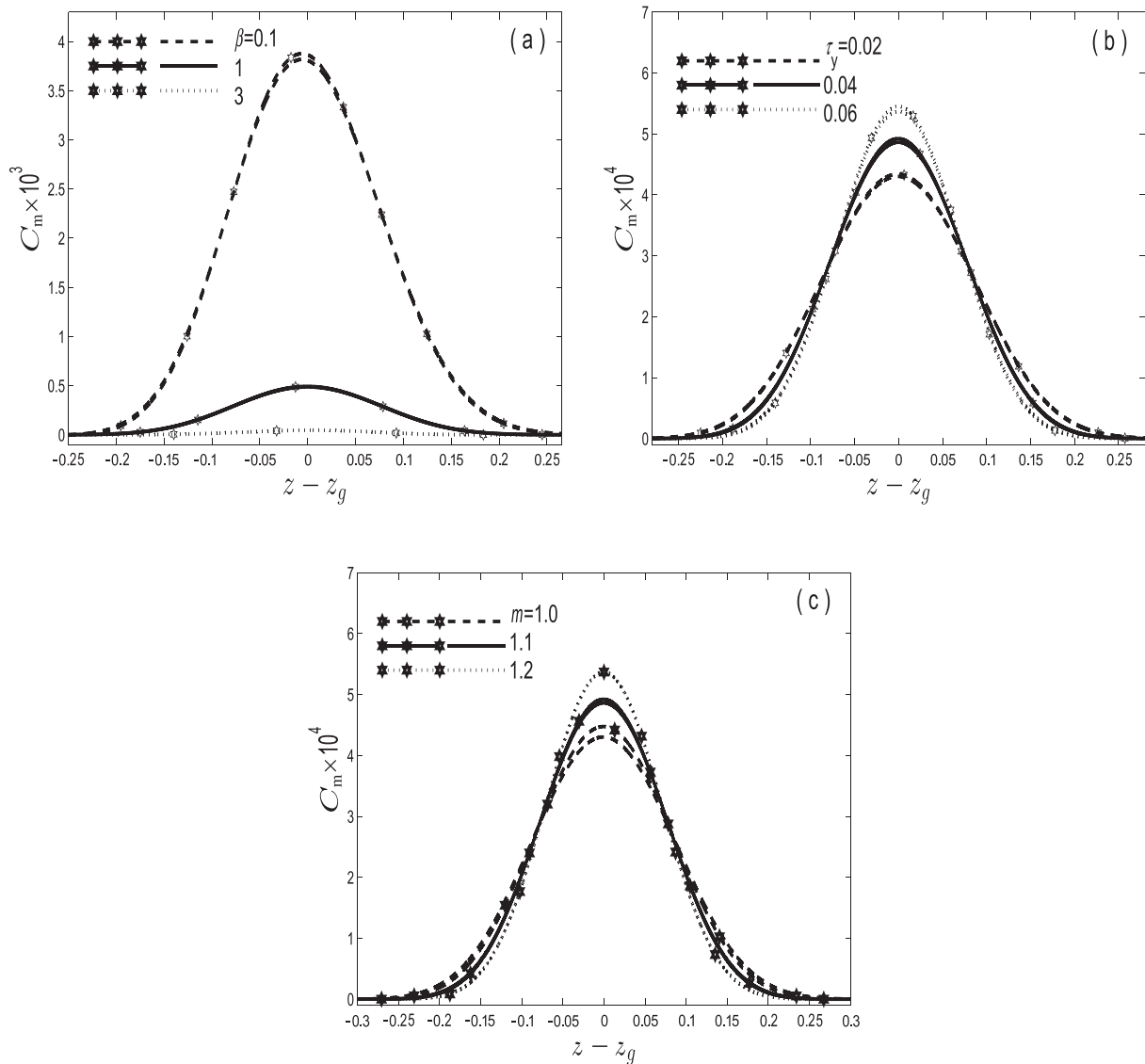


Fig. 8. Variation of mean concentration distribution, at a given instant $t = 1.5$ for Casson liquid (asterisk lines), $\tau_y = 0.04$, $R_0 = 1$ and layered liquid, $\tau_y = 0.04$, $R_0 = 0.94$. Other descriptions are as in Fig. 7.

both M_1 and D_a increase more in the case of the three-layer liquid stream as compared to that in the Casson liquid environment. From the investigation, in Fig. 7(e, f), it can be calculated that for the time interval $[0, 10]$ and viscosity ratio ($m = 1.1$), the value of M_1 increases in the range $(0.0365\text{--}0.0374)$ whereas the value of D_a increases in the range $(1.49 \times 10^{-6}\text{--}4.5 \times 10^{-5})$.

7.3. Mean concentration profile

Fig. 8 presents the study of mean concentration distribution across axial distance for steady nature of stream at a fixed time $t = 1.5$. Fig. 8(a) deals about the effect of absorption parameter (β) on mean concentration distribution whereas Fig. 8(b) and (c) measures the influence due to τ_y and m respectively. Due to constant depletion occurring at the tube wall in both the cases, the peak of the mean concentration decreases as β increases. It may become flat for large absorption, thus the dispersion coefficient may not have practical importance for large β . While observing Fig. 8(b, c), it is found that as τ_y and m increase, the peak of the mean concentration distribution also increases. From Fig. 8(a, b, c), it can be clearly observed that peripheral layer presence makes faster decrements in peak length as compared to purely Casson liquid flow.

7.4. Effect of peripheral layer on transport coefficients

To observe the effect of peripheral layer variation on the three transport coefficients as well as on mean concentration distribution, a

discernible impact has been observed through Figs. 9 and 10. The exchange coefficient (M_0) is independent of R_o , which has been shown in Fig. 9(a) and it completely agrees on the physics of the present model. Fig. 9(b, c) depicts that the convection coefficient (M_1) and dispersion coefficient (D_a) both decrease as the value of R_o increases (or peripheral layer thickness decreases). The physical explanation about such kind of behavior is that the larger value of R_o makes the smaller region for Newtonian liquid, hence the velocity of the liquid decreases resulting in the smaller convection and diffusion rate. Fig. 10 clearly indicates that peak of the mean concentration distribution increases as R_o increases. This is due to slow tracer spreading over the entire region.

7.5. Effect of molecular diffusivity on dispersion coefficient

The molecular diffusivity or diffusion coefficient (D) is a kinematic property of the fluid which implies that the mass of the substance diffuses through a unit surface in a unit time at a concentration gradient of unity. So, diffusion coefficient (D) represents the spreading process, the larger value of D indicates the rapid spreading of the tracer curve whereas smaller D implies slow spreading of solute in the liquid. Fig. 11 shows the effect of D on the apparent dispersion coefficient (D_a).

Fig. 11(a–c) clearly shows that larger values in diffusivity belonging from the region R_c or plug region R_p , cause decreases in dispersion coefficient coefficient decreases. The reason for this decrement is due to the fast spreading of molar flux which completely agrees on the physics of the system. Fig. 12 demonstrated the mean concentration distribution

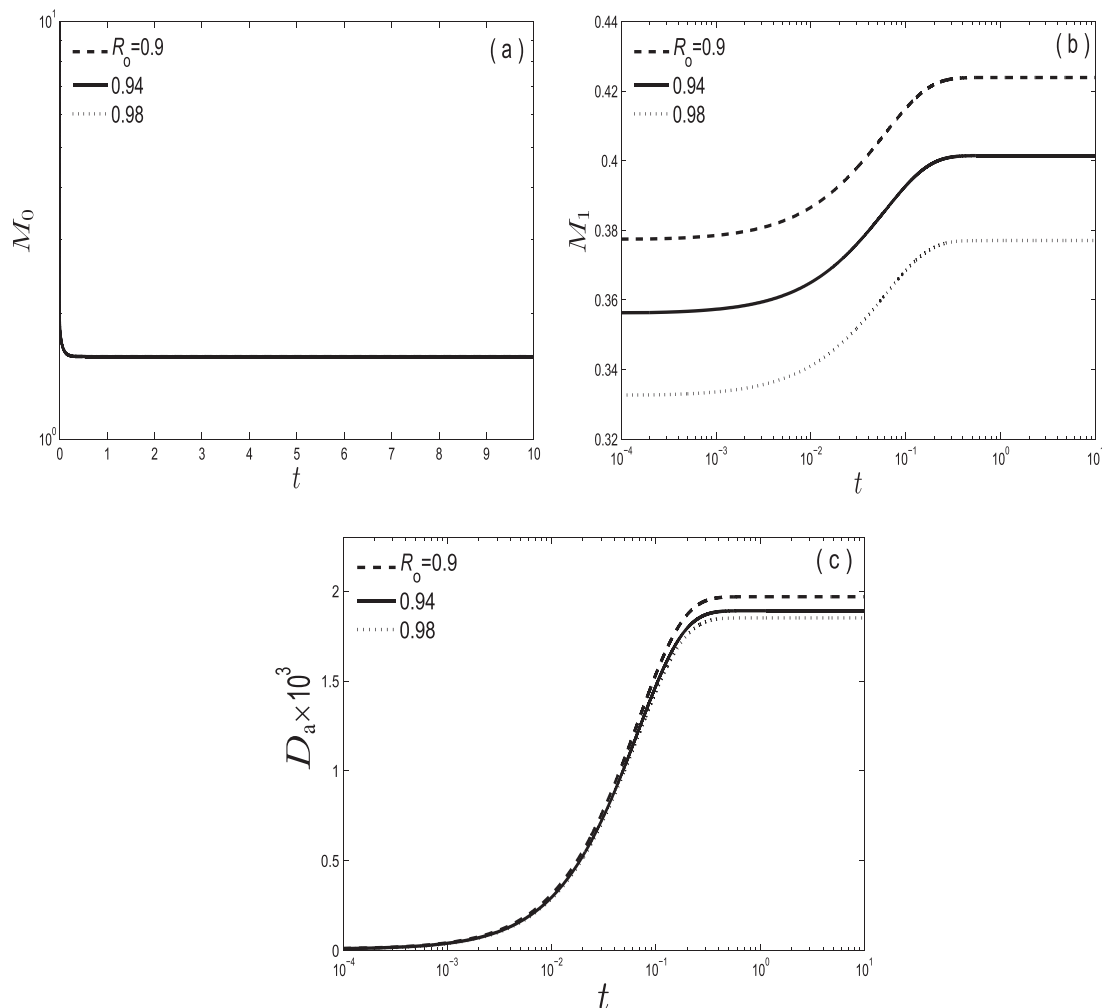


Fig. 9. Time assessment of transport coefficients (M_0 , M_1 and D_a) for various values of R_o for the case when $\tau_y = 0.04$, $m = 1.1$, $\beta = 1$, $D_p = D_c = 1$.

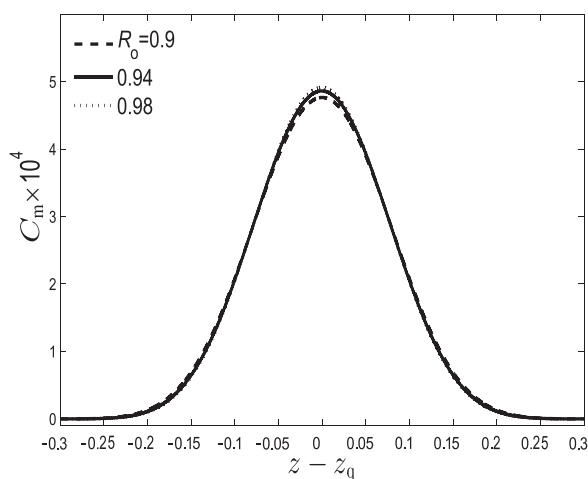


Fig. 10. Variation of mean concentration distribution, at a given instant $t = 1.5$ for the case when $\tau_y = 0.04$, $m = 1.1$, $\beta = 1$, $D_p = D_c = 1$.

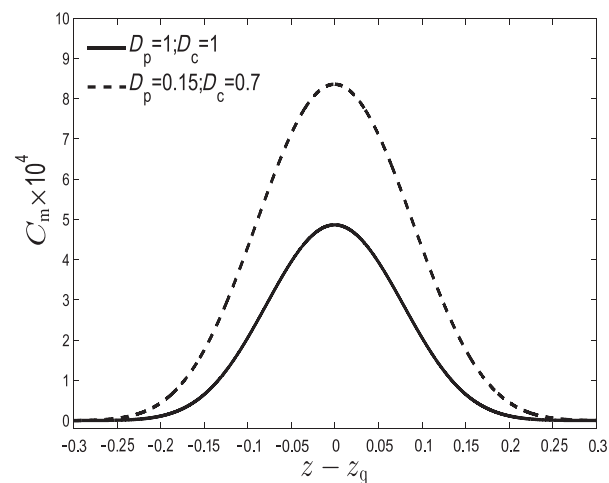


Fig. 12. Variation of mean concentration distribution, at a given instant $t = 1.5$ for the case when $\tau_y = 0.04$, $R_o = 0.94$, $m = 1.1$, $\beta = 1$.

against the axial distance. As evident from Fig. 12, larger diffusivity makes continuous spreading of solute in the radial direction, resulting in a decrease in peak height of mean concentration.

The study of dispersion facilitates in understanding the transport of nutrients in blood and various artificial devices [45,46]. The investigation of dispersion under the proposed model can be deemed to provide

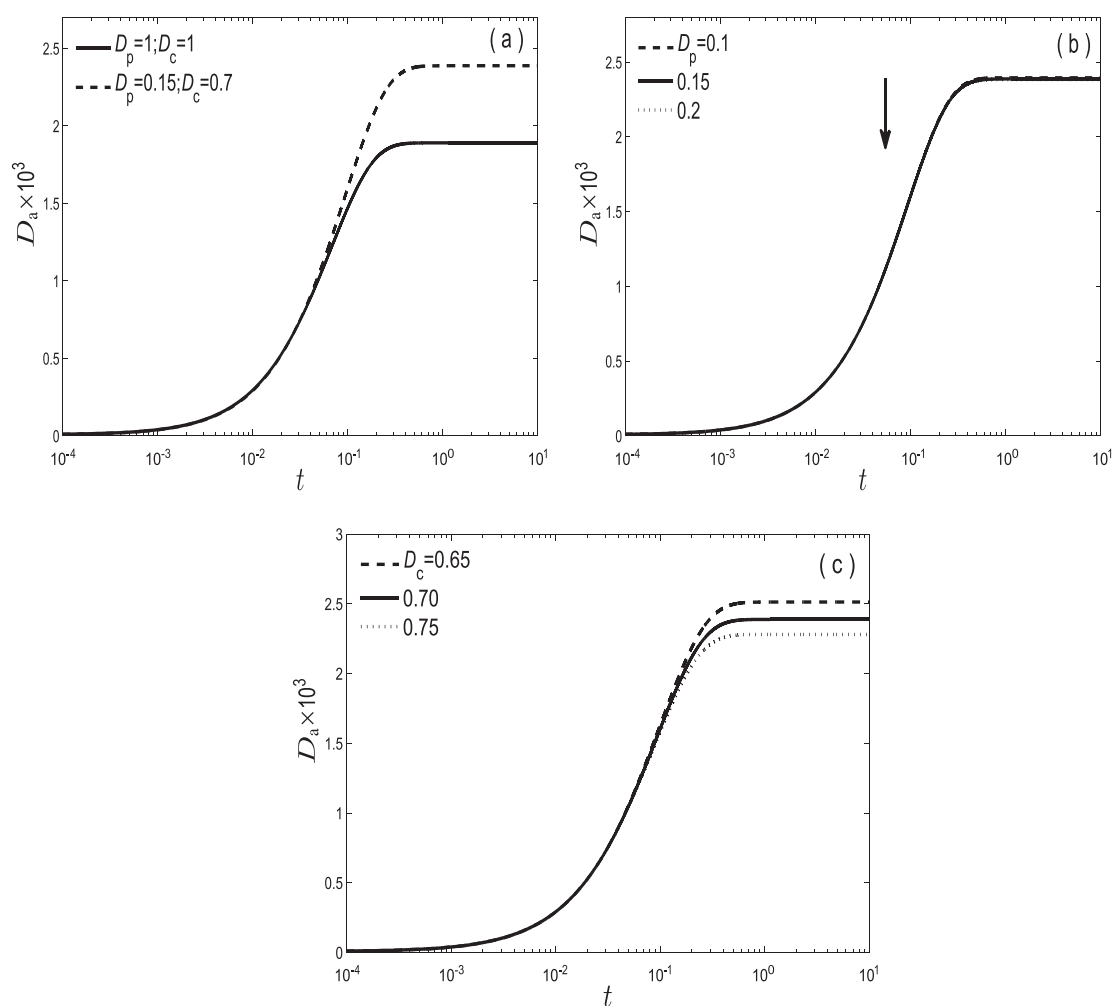


Fig. 11. Variation of dispersion coefficient (D_a) with time for various values of diffusivity for the case when $\tau_y = 0.04$, $R_o = 0.94$, $m = 1.1$, $\beta = 1$. (b) for $D_c = 0.7$ (c) for $D_p = 0.15$.

an insight to comprehend the pathological conditions in the cardiovascular system. Our main objective is to understand the physical mechanism of contaminant mixing or to provide a theoretical framework to interpret the spreading of contaminant in a steady blood-like liquid flow with irreversible wall reaction. Mathematical study on blood flow is greatly supported by the Casson model and when the peripheral layer comes into picture it is more convenient for the analysis. The plug flow region within the Casson liquid region is very important as the significant advantages of transporting particulate materials in a plug form are low energy consumption, low particle attrition and low pipe erosion [47]. Because of the presence of a peripheral layer, different diffusivity has to be considered and a new parameter viz., viscosity ratio appears in the study of solute dispersion which is very important for the studies related to morphology [48]. The results of the present study clearly establish the rightful significance to understand the correlation between the flow and the dispersion in an arterial blood flow situation.

8. Conclusions

From the previously stated discourse, it can be readily concluded that the utility of the updated scientific model described in this paper has a significance in the study of solute dispersion. For all time analysis, the Aris–Barton ‘method of moment’ has been performed on the time developed transport coefficients qualitatively and quantitatively, where a two-liquid model has been considered through a rigid pipe under catalytic wall reaction. Again, the ‘generalized dispersion model’ of Sankarasubramanian and Gill has been employed and the results obtained from both methods are identical, which ensures the authenticity of the results obtained in this study. Mathematical studies on blood flow were mostly studied by the Casson model, but the presence of a peripheral layer on blood has already been accomplished experimentally by different authors. The importance of a peripheral layer on blood flow has been mentioned by few authors while studying resistance to flow, wall shear stress, plug core radius etc. So, the study of dispersion under such kind of situations has an important physical acceptance.

The following general conclusions can be made from the study:

- (i) For all time period, the magnitude of exchange and convection coefficient increases for increasing absorption rate at the pipe boundary though larger absorption makes smaller dispersion coefficient and reaches to its steady value in a short duration of time.
- (ii) The exchange coefficient is independent of the yield stress, viscosity ratio, and peripheral layer thickness.
- (iii) Whatever be the solvent (purely Casson liquid or three-layer liquid), the values of convection and dispersion coefficients always decrease as yield stress and viscosity ratio increase.
- (iv) The presence of a peripheral layer results in a rise in the values of convection and dispersion coefficients as compared to single layer Casson liquid flow.
- (v) Stronger absorption leads to flat the mean concentration distribution and also the peak of the mean concentration distribution increases when yield stress and viscosity ratio increase.
- (vi) The convection and dispersion coefficients both decrease as the peripheral layer thickness decreases.
- (vii) As diffusivity increases, dispersion coefficient decreases, consequently the peak of the mean concentration distribution also goes downward.
- (viii) The findings of the study are also in agreement with those of existing works.

Future Scope of Study

The parameters included in the present scope of study have been singularly motivated to emulate the actual nature of the

physiological situation arising in blood flow analysis as far as possible from a mathematical perspective. However actual physiological conditions attributing to the case study can be further varied to enhance the fidelity of the established model in further research.

Nomenclature

B	variable defined by Eq. (13)
C	concentration of the solute
C_0	initial concentration of the solute
C_m	mean concentration of the solute
$C^{(p)}$	p th moment of the distribution of solute
D^*	molecular diffusion coefficient
D_a	apparent dispersion coefficient for moment method
D_c	molecular diffusivity for Casson flow region
D_n	molecular diffusivity for Newtonian flow region
D_p	molecular diffusivity for plug flow region
f_i	i th dispersion function ($i = 0, 1, 2, 3, \dots$)
H_i	Hermite polynomials
M_0	exchange coefficient
M_1	convection coefficient
M_2	apparent dispersion coefficient for generalized dispersion model
m	viscosity ratio
Pe	Peclet number
R	radius of pipe
R_o	ratio of the central core radius to the normal pipe radius
R_p	plug core radius
r	radial coordinate
t	time
u	axial velocity along the flow
u_0	center line velocity
u_c	axial velocity for Casson flow region
u_n	axial velocity for Newtonian flow region
u_p	axial velocity for plug flow region
z	axial coordinate
z_g	centriod
β	wall absorption parameter
δ_{ij}	Kronecker delta
θ	angular coordinate
μ_c	viscosity of the Casson liquid
μ_n	viscosity of the Newtonian
μ_p	p th order central moment
ν_2	skewness
ν_3	kurtosis
τ_c	shear stress of the Casson liquid
τ_n	shear stress of the Newtonian liquid
τ_y	yield stress
ψ	variable defined by Eq. (13)
$\mathcal{R}_c$	($0 \leq r \leq R_o$) Casson liquid region
$\mathcal{R}_n$	($R_o \leq r \leq 1$) Newtonian liquid region
$\mathcal{R}_p$	($0 \leq r \leq R_p$) plug flow region within Casson liquid region
$\frac{\partial p}{\partial z}$	pressure gradient

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Appendix. Derivation of velocity profile in the three-layer region

The dimensionless momentum forms of Eqs. (1)–(4) are:

$$\frac{\partial(r \tau_c)}{\partial r} = 4r \quad 0 \leq r \leq R_0, \quad (A1)$$

$$\frac{\partial(r \tau_n)}{\partial r} = 4r \quad R_0 \leq r \leq 1, \quad (A2)$$

$$\left. \begin{aligned} \tau_c^{\frac{1}{2}} &= \tau_y^{\frac{1}{2}} + \left(-m \frac{\partial u_c}{\partial r} \right)^{\frac{1}{2}}, & \tau_c \geq \tau_y & \text{for } R_p \leq r \leq R_0, \\ \frac{\partial u_c}{\partial r} &= 0, & \tau_c \leq \tau_y & \text{for } 0 \leq r \leq R_p, \\ \tau_n &= -\frac{\partial u_n}{\partial r}, & \tau_y &= 0 \text{ for } R_0 \leq r \leq 1, \end{aligned} \right\} \quad (A3)$$

$$\left. \begin{aligned} \tau_c &\text{ is finite and } \frac{\partial u_c}{\partial r} = 0 & \text{at } r = 0, \\ \tau_c &= \tau_n \text{ and } u_c = u_n & \text{at } r = R_0, \\ u_n &= 0 & \text{at } r = 1, \end{aligned} \right\} \quad (A4)$$

Solving the boundary value problem of Eqs. (A1)–(A4)

$$\tau_n = 2r \quad (A5)$$

$$u_n(r) = (1-r^2) \quad R_0 \leq r \leq 1, \quad (A6)$$

$$\tau_c = 2r \quad (A7)$$

$$u_c(r) = (1-R_0^2) + \frac{1}{m} \left\{ (R_0^2-r^2) - \frac{4\sqrt{2\tau_y}}{3} (R_0^3-r^3) + \tau_y(R_0-r) \right\} \quad R_p \leq r \leq R_0, \quad (A8)$$

The plug flow velocity counterpart $u_p(r)$ can be obtained by taking $r = R_p$ in Eq. (A8) as:

$$u_p(r) = (1-R_0^2) + \frac{1}{m} \left\{ (R_0^2-R_p^2) - \frac{4\sqrt{2\tau_y}}{3} (R_0^3-R_p^3) + \tau_y(R_0-R_p) \right\} \quad 0 \leq r \leq R_p, \quad (A9)$$

Eqs. (A9), (A8), and (A6) are the velocity distributions in the three-layers documented in Eq. (16).

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